3.15. Conditionals and Biconditionals: Deductive Rules

1. Conditional Rules. In the previous chapter we coupled a general deductive strategy – using ID all cases except those simple enough not to need it – with an array of inference rules which followed a usefulness ranking. Specifically: we reach first for Elim Rules (and of them, $\wedge$– first, $\vee$– and ~– afterward), applying Intro Rules only to free up deductive logjams.

With conditionals now in hand we revise each part of that strategy. Our default choice is now to use CD to deduce a conditional conclusion, and ID for all other types – unless, in each case, we see a simpler way of reaching the conclusion through inference rules alone.

Concerning inference rules, note first that CD spares us the need of an Intro rule for conditionals. But to capture all the valid arguments in the Chapter Three language we must add two **Elim rules** – following tradition in using Latin labels (rather than the more generic "$\rightarrow$–").[1]

Modus Ponens (MP)	**Modus Tollens (MT)**
$\begin{array}{c} (\bullet \rightarrow \blacktriangle) \\ \bullet \\ \hline \blacktriangle \end{array}$	$\begin{array}{c} (\bullet \rightarrow \blacktriangle) \\ \sim\blacktriangle \\ \hline \sim\bullet \end{array}$

[1] Following more recent logical tradition, we use abbreviated forms of the Latin names for these rules: "Modus Ponens" (rather than the original "Modus Ponendo Ponens") and "Modus Tollens" (rather than "Modus Tollendo Tollens").

English examples illustrate the intuitive appeal of each rule.

1. If Rex is home, the light is on.	1. If Rex is home, the light is on.
2. Rex is home.	2. (But) The light isn't on.
∴ The light is on.	∴ Rex (must) not be home.[2]

As with earlier Elim rules, **MP and MT** are to be **used whenever possible**. The following example illustrates how MP and MT (plus an earlier Elim rule) can back us into the desired sentence.

1. If Neko has tuna, then if she has bread she can make a sandwich.
2. Neko has tuna, but she (still) can't make a sandwich.

∴ Neko (must) not have bread.

1. $(P \rightarrow (Q \rightarrow R))$
2. $(P \wedge {\sim}R)$

Get: $\sim Q$

3.	P	2, $\wedge-$
4.	$\sim R$	2, $\wedge-$
5.	$(Q \rightarrow R)$	1, 3, MP
6.	$\sim Q$	4, 5, MT

2. Biconditional Rules. We saw that a biconditional sentence is logically equivalent to the conjunction of a conditional and its converse – e.g., that "$(P \leftrightarrow Q)$" is equivalent to "$((P \rightarrow Q) \wedge (Q \rightarrow P))$".[3] Our Intro and Elim rules for biconditionals reflect this, being really just Conjunction Intro and Elim in disguise.

[2] Conclusion-marking "must" is added here simply to make the example flow more naturally in English.
[3] In 3.4.

Biconditional Introduction (↔+)

$$\frac{\begin{array}{c}(\bullet \to \blacktriangle)\\(\blacktriangle \to \bullet)\end{array}}{(\bullet \leftrightarrow \blacktriangle)} \qquad\qquad \frac{\begin{array}{c}(\blacktriangle \to \bullet)\\(\bullet \to \blacktriangle)\end{array}}{(\bullet \leftrightarrow \blacktriangle)}$$

Biconditional Elimination (↔−)

$$\frac{(\bullet \leftrightarrow \blacktriangle)}{(\bullet \to \blacktriangle)} \qquad\qquad \frac{(\bullet \leftrightarrow \blacktriangle)}{(\blacktriangle \to \bullet)}$$

Since a biconditional is equivalent to a conjunction of two conditionals (one the converse of the other), when a biconditional is needed it can reached by first deducing the two conditionals, and then inferring the biconditional through ↔**+**. For example, to deduce “(P ↔ Q)” from some premises, first deduce “(P → Q)” and “(Q → P)” using CD, then derive the biconditional by **(**↔+**).**

Deduction Strategy: To deduce a biconditional, deduce the corresponding **conditional and its converse**, then use **Bicon Intro (↔+)**.

Likewise, whenever we have a biconditional we know that the corresponding conditional and converse are deducible from ↔**−** (just as, whenever we have a conjunction, we know its left and right parts are deducible by ∧**−**). So, as with all Elim rule, we use ↔**−** whenever possible, with no forethought as to the usefulness of the conditionals this yields us.

Deduction Strategy: Use the Elim Rules MP, MT, and Bicon Elim **(**↔**−**) whenever possible.

Summary

Conditional and Biconditional Rules:

- **Modus Ponens** (**MP**), **Modus Tollens** (**MT**), and **Biconditional Elimination** (**↔−**) are Elim rules, and should be used automatically whenever possible.
- **Biconditional Introduction** (**↔+**) is an Intro rule, and should be used only to yield a specific sentence needed to complete a deduction or to set up an Elim rule.

Deductions Involving Conditionals and Biconditionals:

- To **deduce a conditional**, automatically **use CD** (unless a simpler way of deducing the conditional is obvious).
- To **deduce a biconditional**, deduce the corresponding **conditional and converse** using CD, then **use Bicon Intro** (**↔+**) (unless a simpler way of deducing the biconditional is obvious).

Inference Rules (Chapter Three)

Disjunction Rules

Disjunction Elimination ($\lor-$)

$$\frac{\begin{array}{c}(\bullet \lor \blacktriangle)\\ \sim\bullet\end{array}}{\blacktriangle} \qquad \frac{\begin{array}{c}(\bullet \lor \blacktriangle)\\ \sim\blacktriangle\end{array}}{\bullet}$$

Disjunction Introduction ($\lor+$)

$$\frac{\bullet}{(\bullet \lor \blacktriangle)} \qquad \frac{\blacktriangle}{(\bullet \lor \blacktriangle)}$$

Conjunction Rules

Conjunction Elimination ($\land-$)

$$\frac{(\bullet \land \blacktriangle)}{\bullet} \qquad \frac{(\bullet \land \blacktriangle)}{\blacktriangle}$$

Conjunction Introduction ($\land+$)

$$\frac{\begin{array}{c}\bullet\\ \blacktriangle\end{array}}{(\bullet \land \blacktriangle)} \qquad \frac{\begin{array}{c}\blacktriangle\\ \bullet\end{array}}{(\bullet \land \blacktriangle)}$$

Negation Rules

Negation Elimination ($\sim-$)

$$\frac{\sim\sim\bullet}{\bullet}$$

Negation Introduction ($\sim+$)

$$\frac{\bullet}{\sim\sim\bullet}$$

Repetition (R)

$$\frac{\bullet}{\bullet}$$

Conditional Rules

Modus Ponens (MP)

(● → ▲)
●
―――――
▲

Modus Tollens (MT)

(● → ▲)
~▲
―――――
~●

Biconditional Rules

Biconditional Elimination (↔−)

(● ↔ ▲)
―――――
(● → ▲)

(● ↔ ▲)
―――――
(▲ → ●)

Biconditional Introduction (↔+)

(● → ▲)
(▲ → ●)
―――――
(● ↔ ▲)

(▲ → ●)
(● → ▲)
―――――
(● ↔ ▲)